

Abstract

We analyze the thermodynamical behavior of black holes in closed finite boxes. First the black hole mass evolution is analyzed in an initially empty box. Using the conservation of the energy and the Hawking evaporation flux, we deduce a minimal volume above which one black hole can loss all of its mass to the box, a result which agrees with the previous analysis made by Page. We then obtain analogous results using a box initially containing radiation, allowed to be absorbed by the black hole. The equilibrium times and masses are evaluated and their behavior discussed to highlight some interesting features arising. These results are generalized to N black holes + thermal radiation. Using physically simple arguments, we prove that these black holes achieve the same equilibrium masses (even that the initial masses were different). The entropy of the system is used to obtain the dependence of the equilibrium mass on the box volume, number of black holes and the initial radiation. The equilibrium mass is shown to be proportional to a *positive* power law of the effective volume (contrary to naive expectations), a result explained in terms of the detailed features of the system. The effect of the reflection of the radiation on the box walls which comes back into the black hole is explicitly considered. All these results (some of them counter-intuitive) may be useful to formulate alternative problems in thermodynamic courses for graduate and advanced undergraduate students. A handful of them are suggested in the Appendix.

Thermodynamics of black holes in finite boxes

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1 Introduction

In a seminal work of the 70's, Hawking [1], showed that black holes are capable of emitting radiation, and evaluated the quantum process by which these black holes lose mass due to vacuum polarization (induced by the gravitational energy of the black hole). This important work and follow-up contributions, together with Bekenstein's arguments [2] about the entropy of black holes, prompted a new discipline of black hole research: thermodynamics of black holes. Today, the study of black hole thermodynamics is very active and several arguments indicate that it must embrace the partial (or perhaps complete) marriage of quantum mechanics, thermodynamics and gravitation.

The original Hawking analysis showed that the particles created display a thermal spectrum $f(E, T) = \frac{1}{e^{\beta E} \pm 1}$ with $\beta = 1/k_B T$, and a temperature determined by the black hole mass M given by

$$T_{bh} = \frac{\hbar c^3}{8\pi k_B GM} \sim 10^{-7} \left(\frac{M_\odot}{M} \right) K \quad (1)$$

see [3] for details. Since this spectrum is thermal, the luminosity of the black hole can be calculated using the Stefan-Boltzmann law, using $L \propto T^4 r_g^2$, where the gravitational radius $r_g \propto M$ squared plays the role of the emitting area. Using these definitions, $L(M) \propto M^{-2}$ is readily obtained.

In addition to "purely academic" cases (e.g. black holes inside ideal boxes), these emission (absorption) relations are in principle useful for investigating the properties of astrophysical and/or primordial black holes (hereafter PBHs) and their cosmological consequences. For example, some short-timescale gamma-ray burst events have been possibly explained by evaporating PBHs in a terminal (explosive) phase. In fact, the derived relations between period, mass and spectra are consistent with the PBH model, see [4].

Even in the simplest cases (the black holes interacting with ambient radiation) it is interesting to understand what determines the equilibrium features of the system and how is it achieved. We shall perform an analysis restricted to finite volumes with the aim of illustrating some novel features of the thermodynamics of black holes using *gedanken* experiments. These exercises serve also as a prelude to the more complicated situation, in which evaporation of PBHs must be studied in an expanding universe. However, and quite independently of these further advanced considerations, "boxed" black holes display interesting features which need careful clarification.

2 Classical absorption in an initially cold box

Let us consider the following *gedanken* experiment: we start with a black hole and put it within a finite box (with linear size L). In the beginning, the box has no radiation, i.e. its temperature is zero. Then, since the black hole is hotter than the environment, it will emit thermal radiation, and thus this closed box will be increasingly filled with this radiation shortly afterwards.

The evolution of the mass of the black hole is described by the well-known expression which balances the energy losses to the Hawking luminosity $\propto M^{-2}$

$$\frac{dM}{dt} = -\frac{C}{M^2} \quad (2)$$

with an initial condition of the box $T_{rad}(t = t_i) = 0$. The constant C depends on the degrees of freedom allowed to be radiated by the black hole [1] and is set to $C \sim 10^{26} g^3 s^{-1}$ throughout this work, and the initial mass is M_i . We assume that the box linear size is much larger than the black hole gravitational radius $L \gg r_g$ to avoid a complicated non-linear feedback between the geometry and the radiation field, which would lead us far beyond

the purely thermodynamical approach.

Since the box is closed, the total energy is conserved. Therefore it is easy determine a relation for the black hole mass and the thermal radiation contained within the box from

$$\varrho_{rad}(T_{rad})V_{box} + M(t) = M_i \quad (3)$$

where $t > t_i$ (we use natural units to set the speed of light c equal to 1). The radiation density is given by $\varrho_{rad}(T) = a_* T^4$ with $a_* \sim 8 \times 10^{-36} g cm^{-3} K^{-4}$. After some time, the box temperature is

$$T_{rad}(t) = \left[\frac{M_i - M(t)}{a_* V_{box}} \right]^{1/4} \quad (4)$$

At an even later time $t > t_i$, eq.(2) should not simply contain the Hawking term, but an absorption term is also needed. The simplest form of term is generally constructed as the product of the incoming flux times the gravitational cross-section of the radiation falling into the hole, $B\varrho_{rad}(T)M^2$, with $B = \frac{27\pi G^2}{c^3}$ (or $\frac{27\pi}{M_{pl}^4}$ in natural units, see Ref. [5] for details and discussion). Therefore the complete differential equation for the mass is given by

$$\frac{dM}{dt} = -\frac{C}{M^2} + B\varrho_{rad}(T)M^2 \quad (5)$$

From this equation, it is clear that thermodynamical equilibrium between the black hole plus and the ambient radiation will be given by $\dot{M} = 0$, and thus the mass of the black hole in equilibrium is

$$M_{bh} = M_c(t_{eq}) = \frac{D}{T_{rad}(t_{eq})} \quad (6)$$

where $D = (C/a_* B)^{1/4} \sim 2 \times 10^{26} g K$. Solving the set of equations above we track the black hole mass evolution as it approaches the equilibrium (and determine the conditions for this equilibrium).

If we substitute the eq.(4) into eq.(5) we obtain

$$\frac{dM}{dt} = -\frac{C}{M^2} + B \left(\frac{M_i - M}{V_{box}} \right) M^2 \quad (7)$$

The evolution of the mass of eq.(7) will be given by solving the integral

$$\int_{M_i}^{M(t)} \frac{dMM^2}{-\kappa_1 + \kappa_2 M^4 + \kappa_3 M^5} = t \quad (8)$$

and the constants κ_i are $\kappa_1 = C$, $\kappa_2 = BM_i/V_{box}$ and $\kappa_3 = -B/V_{box}$.

In the limit $V_{box} \gg r_g^3$ the solution of eq.(7) approaches $M(t) \sim M_i[1 - (t/t_{evap})]^{1/3}$ with $t_{evap} = \frac{M_i^3}{3C}$, as expected.

The numerical solution of the integral eq.(7) is shown in Fig.1 for different box sizes but the same initial black hole mass M_i . In order to obtain the conditions for the equilibrium we solve the algebraic equation $T_{rad}(t_{eq}) = T_{Haw}(M_{eq})$, rewritten as

$$\left[\frac{M_i - M(t_{eq})}{a_* V_{box}} \right]^{1/4} = \frac{C}{M(t_{eq})} \quad (9)$$

Inserting the respective constants above, eq.(9) becomes

$$[1 - \mu(t_{eq})]\mu^4(t_{eq}) = D(V_{box}/cm^3) \times (M_\odot/M_i)^5 \quad (10)$$

where $\mu(t_{eq}) = \frac{M(t_{eq})}{M_i}$ and $D \sim 4.2 \times 10^{-97}$.

The interpretation of the solution is quite evident, when the volume of the box is such that the right side of eq.(9) is larger than one, the black hole evaporates completely in the absence of additional quantum corrections at the Planck scale. But if $D(V_{box}/cm^3) \times (M_\odot/M_i)^5 \sim 1$, the size of the box is small enough to *stop* the evaporation, and the system black hole+radiation achieves thermodynamical equilibrium before its mass vanishes completely.

The linear size of the box below which the black hole achieves its equilibrium is given by

$$L_c(M_i) \sim 1.3 \times 10^{32} (M_i/M_\odot)^{5/3} cm \quad (11)$$

The time for reaching equilibrium t_{eq} is determined by the initial mass and the size of the box only, i.e. $t_{eq} = t_{eq}(M_i, V_{box})$. For the same initial mass, smaller boxes will contain black holes with larger final masses at equilibrium (see Fig. 1). Our analysis agrees with the previous work by Page [7]. From now on, we define the *critical volume* by $V_{crit}(M) = L_c(M)^3$ using eq.(10), with its clear physical interpretation given above. In the next sections we revisit the derivation of eq.(10), extend it to more complicated cases and consider causality features.

Figure 1: The approach to equilibrium of black holes in finite boxes. The curves represent qualitatively the temporal behavior of the mass of the black hole for two different values of the box size V_{box} (initially devoid of radiation); with the initial value of the mass M_i held fixed.

3 Classical absorption in a closed box with initial radiation

Let us now evaluate the behavior of the black hole mass when introduced in a box with a non-zero initial radiation content. We expect the black hole to achieve thermodynamical equilibrium earlier than in the case without initial radiation, and with a smaller equilibrium mass (considering boxes with the same size). In this case, is easy show that

$$\int_{M_i}^{M(t)} \frac{dMM^2}{-\kappa_1 + \kappa_4 M^4(E_i - M)} = t \quad (12)$$

where $E_i = M_i + a_* T_i^4 V_{box}$ and $\kappa_4 = B/V_{box}$. The plot of $M(t)$ as a function of time is actually similar to the former case.

As before, we impose the equilibrium condition in to obtain the relation between the black hole mass and the box volume. Following the same steps as before, we obtain

$$\mu^*(t_{eq})^4 [F(T_i, M_i) - \mu^*(t_{eq})] = D(V_{box}/cm^3) \times (M_\odot/M_i)^5 \quad (13)$$

where $F(T_i, M_i) = \frac{E_i}{M_i}$, and the t_{eq} is different from the previous case. Actually the black hole achieves thermodynamical equilibrium sooner, as expected.

4 N black holes plus radiation in a closed box

The generalization to N black holes seems quite straightforward, although it will become clear that the initial states and other details must be carefully defined to achieve consistent results. First, we shall consider just two black holes immersed in a closed box (with constant volume) in the following initial

situation: one black hole has initial mass M_1 and the other black hole is more massive than the first, $M_2 > M_1$. Initially the box has no radiation, and therefore these objects begin to evaporate immediately. Moreover, we shall consider that $V_{box} \sim V_{crit}(M_1)$. Then at the initial time t_i we have

$$\left(\frac{dM_2}{dt}\right)_{t_i} = -\frac{C}{M_2^2} \quad (14)$$

$$\left(\frac{dM_1}{dt}\right)_{t_i} = -\frac{C}{M_1^2} \quad (15)$$

Since $M_1 < M_2$ one may consider $|(\dot{M}_1)_{evap}| \gg |(\dot{M}_2)_{evap}|$ initially. Afterwards, when the box starts to be filled with the emitted radiation, some energy can be absorbed by the black holes. Therefore, their evolution will be given by

$$\frac{dM_2}{dt} = -\frac{C}{M_2^2} + B\varrho_{rad}(T)M_2^2 \quad (16)$$

$$\frac{dM_1}{dt} = -\frac{C}{M_1^2} + B\varrho_{rad}(T)M_1^2 \quad (17)$$

$$M_1(t) + M_2(t) + V_{box}\varrho_{rad}(T) = M_1(t_i) + M_2(t_i) \quad (18)$$

where the last condition displays the conserved energy of the box. Using this constraint, we can describe this system by the following set of equations

$$\frac{dM_2}{dt} = -\frac{C}{M_2(t)^2} + \frac{B}{V_{box}}M_2(t)^2[E_i - M_1(t) - M_2(t)] \quad (19)$$

$$\frac{dM_1}{dt} = -\frac{C}{M_1(t)^2} + \frac{B}{V_{box}}M_1(t)^2[E_i - M_1(t) - M_2(t)] \quad (20)$$

combining eqs.(17) and (18) and using the conservation of energy yields

$$-C\left[\frac{1}{M_1^2} + \frac{1}{M_2^2}\right] + B\varrho_{rad}[M_1^2 + M_2^2] + V_{box}\dot{\varrho}_{rad} = 0 \quad (21)$$

where $E_i = M_1 + M_2$ is the initial energy.

Note that the set of coupled equations does not include the effect of black hole motion due to their mutual gravity. Motion is likely to affect the

emission/absorption properties of the black holes in the relativistic regime [8], [9]. Thus, the analysis is strictly valid whenever $v_{bh} \ll c$.

Let us take a look at the global behavior of the solutions. First, it is easy to show that the equilibrium between two black holes is possible. If we impose that the thermodynamical equilibrium will be achieved, then asymptotically

$$\frac{dM_2}{dt} = \frac{dM_1}{dt} = 0 \quad (22)$$

From these equations $\frac{C}{B}V_{box} = M_1^4[E_i - M_1 - M_2]$ immediately follows, and the same is true for the other black hole. Therefore, it follows that

$$M_{1,eq} = M_{2,eq} \quad (23)$$

The same results of these equilibrium masses could have been obtained from the expression of the entropy. In fact, more complicated cases can be worked out either using the above approach or using that the entropy must be an extreme for a system in equilibrium. For instance, let us consider N black holes enclosed in the box. The total entropy for this system is given by

$$S_{total}(t) = \frac{1}{4} \sum_i A_i(t) + g_* T_{rad}^3(t) V_{box} \quad (24)$$

Since the horizon area for the i -th black hole is $A_i = 4\pi r_{g,i}^2$ we have

$$S_{total}(t) = \frac{4\pi}{M_{pl}^4} \sum_i M_i(t)^2 + g_* T_{rad}^3(t) V_{box} \quad (25)$$

In equilibrium $T_{rad} = T_{bh1} = T_{bh2} = \dots = T_{bhi} = \frac{M_{pl}^2}{8\pi M_{eq}}$, and therefore

$$S_{max}(N, V_{box}) = \beta_* N \mu_{eq}^2 + l_* / \mu_{eq}^3 (V_{box}/cm^3) \quad (26)$$

where $\beta_* = 2.5 \times 10^{40}$, $l_* = 8 \times 10^{33} \alpha_*$ and $\mu_{eq} = (M_{eq}/10^{15} g)$ the equilibrium mass scaled to a convenient reference value.

Extremizing the entropy $\left(\frac{dS_{max}}{dM_{eq}} \right) = 0$ and solving for M_{eq} yields

$$M_{eq}(N, V_{box}) \sim 5.4 \times 10^{13} \alpha_*^{1/5} N^{-1/5} (V_{box}/cm^3)^{1/5} g \quad (27)$$

with $\alpha_* = 26.7 g_*$. After substituting back into S_{max} we find

Figure 2: Graphical solution of the non-linear equation for the equilibrium mass (eq.30) as a function of time for a fixed value of V_{box} and a different number of black holes N . It may be said that the equilibrium mass "feels" the presence of other black holes.

$$S_{max}(N, V_{box}) \sim 1.2 \times 10^{38} \alpha_*^{2/5} N^{3/5} (V_{box}/cm^3)^{2/5} \quad (28)$$

Since $N = nV_{box}$ by definition, we have $S_{max} \sim 1.2 \times 10^{38} n^{3/5} (V_{box}/cm^3)$. Therefore, the thermodynamic approach leads us to conclude that the equilibrium black hole mass is bigger for bigger boxes. This result is quite counter-intuitive, but it has a reasonable explanation given below. To confirm the dependence first, we resort to another method.

Let us rewrite the equation for the mass of the i -th black hole as

$$\frac{dM_i}{dt} = -\frac{C}{M_i(t)^2} + \frac{B}{V_{box}} M_i(t)^2 [E_i - NM_i(t)] \quad (29)$$

where we prepare the system with N black holes with the same initial mass (this particular condition is not over-restrictive and simplifies the algebra). From $\dot{M}_i = 0$ we obtain

$$\mu_{eq}^4 - \frac{1}{\mu_i} \mu_{eq}^5 = \frac{K}{N\mu_i} (V_{box}/cm^3) \quad (30)$$

where $K \sim 3.5 \times 10^{-6}$. This algebraic equation has non-trivial solutions which can be found numerically. For the sake of definiteness we adopt $N = 1$, $\mu_i = 10$, and a small $V_{box} = 10^3 cm^3$, which yields $\mu_{eq} \sim 0.137$. If we enlarge the box, say $V_{box} = 10^4 cm^3$ instead, then $\mu_{eq} \sim 0.244$, and so on. If we fix the size of the box instead, say $V_{box} = 2.8 \times 10^5 cm^3$, and change the number of black holes, the equilibrium mass will reflect the effect of the presence of other black holes accordingly. For example, taking $\mu_i = 10$ and $N = 1$ we have $\mu_{eq} \sim 1.027$ but for $N = 10^3$ we obtain $\mu_{eq} \sim 0.178$ (Fig. 2). These results agree with the previous entropy analysis, and confirm that the coupled equations correctly describe the evolution of the system. It is quite interesting to propose the construction of a plot showing the behavior of the solutions and a discussion about the reasons for that behavior.

As a further related work it is proposed to show that if we consider the existence of an initial thermal radiation with energy E_i , the effective volume becomes

$$V_{eff}(N, M_i) = \frac{V_{box}}{N + E_i/M_i} \quad (31)$$

which holds for N black holes with the same initial masses. This effective volume must be used instead of (V_{box}/N) in eq.(26) and $M_{eq}(N, V_{box})$ becomes $\propto [\frac{V_{box}}{N+E_i/M_i}]^{1/5}$.

The puzzling growth of the equilibrium mass in these systems must be further explained in more detail to improve the understanding. To proceed, let us consider two boxes with volumes $V_{box,2}$ and $V_{box,1}$ where the second box is larger than the first. Let us immerse two black holes with the same initial masses in both boxes. Therefore, the energy content is forced to be the same. Then, in equilibrium we will have (dropping some inessential numerical factors)

$$E_{box} = 2M_{eq,1} + T_{rad,1}^4 V_{box,1} = 2M_{eq,2} + T_{rad,2}^4 V_{box,2} \quad (32)$$

Since the equilibrium mass is given by $M_{eq} \propto [V_{box}]^{1/5}$ for both boxes, then we have

$$V_{box,1}^{1/5} - V_{box,2}^{1/5} = T_{rad,2}^4 V_{box,2} - T_{rad,1}^4 V_{box,1} \quad (33)$$

both terms are negative due to the assumption $V_{box,2} > V_{box,1}$ then we obtain from the r.h.s. of this equation

$$\left(\frac{T_{rad,2}}{T_{rad,1}} \right)^4 < \left(\frac{V_{box,1}}{V_{box,2}} \right) < 1 \quad (34)$$

then $T_{rad,2} < T_{rad,1}$. Therefore, the bigger box will have smaller radiation temperature. Since in equilibrium $T_{rad,2} = T_{bh,2} \propto \frac{1}{M_{eq,2}}$ then the mass of the second black hole must be bigger than the first, in agreement with our previous analysis. Previous work by Page [7] (specifically his eq.(10)) has explored this problem, which has been solved here using the conservation of energy only, with the absorption terms explicitly present.

5 Causality considerations

The above considerations implicitly assumed that the black hole absorption starts instantaneously as the former is created inside the box. One might wonder whether a black hole immersed within an initially empty box is actually able to absorb ambient radiation immediately; since the Hawking radiation emitted by it will be available to absorption only after $\sim L/c$ as required by causality.

In order to gain some insight, we use the fact that if the box is large enough, when the radiation comes back into the black hole, the object has evaporated completely. Then, a critical condition is obtained imposing this time interval to be bigger than the timescale for evaporation. Thus

$$\frac{L}{c} > \frac{M^3}{3C} \quad (35)$$

Therefore, it follows that $L_{crit}(M) \sim 10^{29} cm \left(\frac{M}{10^{15} g} \right)^3$.

Then, any box (without initial radiation) containing a black hole with mass M whose linear size L is bigger than the critical size L_{crit} above would let the black hole to evaporate completely, thus precluding equilibrium. Note that the functional dependence and numerical value are very different from the naive approach used previously where we have considered that the absorption term turns on instantaneously (that is, as soon we immersed the black hole in the box). If we boldly compare L_{crit} with the size of the Hubble horizon today $\sim 10^{27} cm$, we deduce that a PBHs with masses smaller than $\sim 2 \times 10^{14} g$ can evaporate completely. However, it is clear that in a realistic case other sources of energy must be considered and our comment is intended just for pedagogical purposes.

The result of eq.(35) can be interpreted as follows. If we put one hot black hole (at t_i) with initial mass M_i inside a closed box with volume V_{box} , it will evolve according to $\dot{M} = -C/M^2$ for a time $t_i < t < \frac{L_{box}}{c}$. But from $t > L_{box}/c$ on, the black hole has to absorb radiation from the box (originated by itself!). Then the absorption term $\propto T^4 V_{box}$ contributes to avoid its complete evaporation. If the box is slightly larger (but smaller than the critical volume defined by eq.(10)) the onset of absorption is delayed, then this black hole will be hotter than the previous case. This black hole is then filling the box with higher temperature radiation, and after some time it will be absorbing

radiation at higher temperature, hence the absorption term $\propto T^4 M^2$ will be larger. This term drives the black hole into the larger equilibrium mass, as given by eq.(27). It should be remembered that a maximal number for black holes enclosed in a box with volume V_{box} must exist, a naive estimate of N_{max} can be obtained from the close-packing condition $N_{max} r_g^3(M_{eq}) \sim V_{box}$.

6 Conclusions

In this work we have analyzed the behavior of black holes+radiation systems, considering some *gedanken* experiments within finite boxes (initially with and without radiation) (see [10] for a clear pioneering discussion of these issues). These *gedanken* experiments are important to understand the evolution of black holes in more complex situations, since it is hoped that we can always approximate the thermodynamical behavior of the universe with that of a finite box. In these simple cases the energy available for absorption by the black holes is finite at all times and there is no need to consider other sources. Strictly speaking, and with an eye on the actual cosmological formation and evolution of black holes, it may be argued that the problem is oversimplified, since actual primordial black hole masses must be formed with a substantial fraction of the horizon mass [6], and therefore finite size effects should be considered from the scratch. The actual situation is much more complicated but not impossible to tackle. The real difficulty, which can be viewed as an advanced exercise dealing with the (generalized) second law of thermodynamics not related to laboratory systems, is that (unlike laboratory gases) the black hole "gas" will have varying masses and therefore a complicated kinetic equation must be used to describe their evolution. This provides a concrete example, possibly realized in nature, of a variable-mass gas needing a deep understanding of statistical mechanics principles for its very formulation. The simplest approach taken here was enough to show that one black hole plus radiation can achieve thermodynamical equilibrium if the box volume is smaller than a critical volume (in agreement with a previous treatment made by Page [7]). Some surprises arise in more general cases where we have two (or more) black holes (plus radiation) within these boxes.

Even in their simplest versions the cases of black holes in boxes are very instructive to analyze and may serve as good issues for a graduate course

of thermodynamics/statistical mechanics (see [11] for a discussion of related formal aspects of black hole thermodynamics) as an alternative to traditional problems dealing with the approach to equilibrium. They pose several questions of deep physical meaning, which are also strongly entangled and sometimes puzzling to interpret.

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7 Appendix

A complementary set of simple problems intended to reinforce the conceptual aspects of black hole evaporation. Except for the problem 4, their mathematical complexity is quite straightforward.

- 1) Using eq.(2), prove that the timescale for evaporation for one black hole is proportional to M^3 . Verify that for one black hole with the solar mass ($M \sim 2 \times 10^{33} g$); $t_{evap}(M) \sim 10^{65} \text{ years}$. Evaluate this for a initial mass $\sim 10^{15} g$ and compare with the age of the universe.
- 2) Verify the expression eq.(6) for the critical mass. Evaluate this number today, considering that $T_{rad}(t_0) \sim 3K$.
- 3) Derive eq.(11). Use for the linear size of the box the present size of the universe. Evaluate the corresponding mass at equilibrium.
- 4) Deduce the set of eqs.(19)-(21) and interpret the solutions graphically.
- 5) Deduce the $M_{eq}(N, V_{box})$ for the cases without initial radiation and with a radiation filling the box. Interpret these results and compare with the cosmological situation plugging representative numbers for the latter.
- 6) Discuss how all these results may be modified if the box expands following a known temporal dependence.



